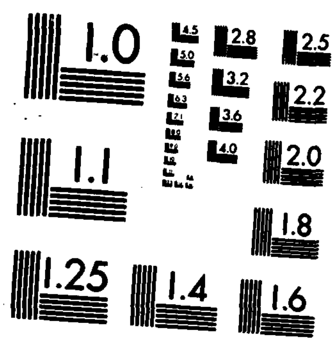

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

NUSC Technical Memorandum 831150
3 October 1983

Resolution of Ambiguity for Randomly Moving Line Array

Albert H. Nuttall
Surface Ship Sonar Department

Naval Underwater Systems Center
Newport, Rhode Island/New London, Connecticut

Approved for public release; distribution unlimited.

TM No. 831150

NAVAL UNDERWATER SYSTEMS CENTER
New London Laboratory
New London, Connecticut

RESOLUTION OF AMBIGUITY FOR RANDOMLY MOVING LINE ARRAY

Date: 3 October 1983

Prepared by: Albert H. Nuttall

Albert H. Nuttall
Surface Ship Sonar
Department

Approved for public release; distribution unlimited

ABSTRACT

The generalized likelihood ratio detector, for deciding between the right-left ambiguity of a line array attempting to estimate the angle of arrival of a plane wave, is derived. Two scenarios are considered, the first with noisy measured antenna angle, the second with noiseless antenna angle measurements. The detector for both cases is a cross-correlator of the sample ac components of the measured antenna and source angle waveforms.

ADMINISTRATIVE INFORMATION

This memorandum was prepared under NUSC Project No. X12012, "Sonar Systems Improvement Program", Principal Investigator B. Spear, Code 3291. The sponsoring activity is Naval Sea Systems Command, PMS 409, AN/BQQ-5 Program Manager J. Archer. Also this memorandum was prepared under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing", Principal Investigator Dr. A. H. Nuttall, Code 3302; Program Manager Capt. Z. L. Newcomb, Naval Material Command, MAT 05B.

The author of this technical memorandum is located at the Naval Underwater Systems Center, New London, CT 06320.

INTRODUCTION

A line array inherently has a cone of ambiguity in its response. When the array lies in the horizontal plane, and a source is located in that same plane, the ambiguity reduces to a right-left uncertainty, which cannot be resolved without some manuevering on the part of the source or array. If the line array is moving randomly, unintentionally or uncontrollably, this movement can serve as a means of making a high quality decision about the source direction, if the array angle, as well as the source angle relative to the line array, are measured.

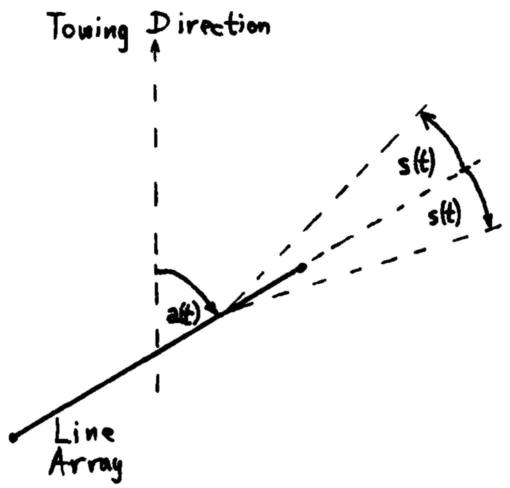

Figure 1. Geometry of Line Array and Source

The situation of interest here is described in figure 1. The line array is being towed due north; however, it is undergoing rigid bar rotation about this direction in a random manner, as described by random process a(t), which is the actual antenna angle relative to the towing direction.

The actual source angle, relative to the line array end-fire direction, is s(t). Furthermore, the actual source angle, relative to the towing direction, is θ, an unknown constant; it is presumed that θ is constant

throughout the observation interval. Reference to figure 1 reveals that these various quantities are interrelated according to the alternatives

$$\theta = \begin{Bmatrix} a(t) + s(t) \text{ for hypothesis 1, } H_1 \\ a(t) - s(t) \text{ for hypothesis 2, } H_2 \end{Bmatrix} . \tag{1}$$

However, it is unknown which hypothesis is correct; nevertheless, it is desired to make a reliable decision, so that an accurate estimate of the source direction can be made. From (1), observe that we can express

$$s(t) = \begin{Bmatrix} \theta - a(t) \text{ for } H_1 \\ -\theta + a(t) \text{ for } H_2 \end{Bmatrix} , \tag{2}$$

which will be needed in later developments.

NOISY MEASURED ANTENNA ANGLE

In this section, the measured antenna angle is not $a(t)$ as desired, but rather is

$$x(t) = a(t) + m(t) , \tag{3}$$

where $m(t)$ is an unavoidable additive noise process. Also the measured source angle is not $s(t)$, but instead is

$$y(t) = s(t) + n(t) , \tag{4}$$

where $n(t)$ is likewise an undesirable additive perturbation, due to limited observation time, array length, ambient noise, etc. The three random processes $a(t)$, $m(t)$, $n(t)$ are presumed to be zero-mean Gaussian processes, independent of each other.

Combining (2)-(4), the situation is as follows: the available measurements upon which a decision must be reached are the two waveforms

$$\left.\begin{aligned} x(t) &= a(t) + m(t) \\ y(t) &= \theta - a(t) + n(t) \end{aligned}\right\} \text{for } H_1 , \tag{5A}$$

or

$$\left.\begin{aligned} x(t) &= a(t) + m(t) \\ y(t) &= -\theta + a(t) + n(t) \end{aligned}\right\} \text{for } H_2 . \tag{5B}$$

On the basis of waveforms $x(t)$ and $y(t)$, what is the best decision and what is the corresponding estimate of θ?

Derivation of Generalized Likelihood Ratio

Let Δ be the time sampling increment applied to measurement waveforms x(t) and y(t); assume that the samples of the three processes are statistically independent at this rate. Denote

$$x_k = x(k\Delta) \text{ for } 1 \le k \le K \,,$$
$$y_k = y(k\Delta) \text{ for } 1 \le k \le K \,, \tag{6}$$

where $K\Delta$ is the total observation time, and let the collections of samples be denoted by

$$X = x_1, x_2, \ldots, x_K, \quad Y = y_1, y_2, \ldots, y_K, \quad A = a_1, a_2, \ldots, a_K \,. \tag{7}$$

Then for a fixed A and a hypothesized value θ_h for the source angle, the conditional probability density function under H_1, of the total set of measurements X, Y, is

$$p_1(X, Y|\theta_h, A) = \prod_{k=1}^{K} \left\{ \frac{1}{\sigma_m} \phi\left(\frac{x_k - a_k}{\sigma_m}\right) \frac{1}{\sigma_n} \phi\left(\frac{y_k - \theta_h + a_k}{\sigma_n}\right) \right\} , \tag{8}$$

where the normalized Gaussian probability density function is

$$\phi(t) = (2\pi)^{-1/2} \exp(-t^2/2) \,. \tag{9}$$

Here we used (5A) and the Gaussian character of processes m(t) and n(t) with standard deviations σ_m and σ_n respectively.

We now must weight (8) by the Gaussian probability density function for process a(t) and integrate over A, to determine the unconditional probability density function of X, Y, for hypothesized value θ_h. Carrying out the integrals and simplifying the result, we obtain

$$p_1(X, Y|\Theta_h) = \left(2\pi\sigma_a\sigma_m\sigma_n\, R_3^{1/2}\right)^{-K} *$$

$$* \exp\left[-\frac{R_{an}}{2R_3\sigma_m^2}\sum_k x_k^2 - \frac{R_{am}}{2R_3\sigma_n^2}\sum_k (y_k-\Theta_h)^2 - \frac{1}{R_3\sigma_m^2\sigma_n^2}\sum_k x_k(y_k-\Theta_h)\right], \tag{10}$$

where we define

$$R_{am} = \frac{1}{\sigma_a^2} + \frac{1}{\sigma_m^2}\,,\ R_{an} = \frac{1}{\sigma_a^2} + \frac{1}{\sigma_n^2}\,,\ R_3 = \frac{1}{\sigma_a^2} + \frac{1}{\sigma_m^2} + \frac{1}{\sigma_n^2}\,, \tag{11}$$

and use the shorthand notation

$$\sum_k = \sum_{k=1}^{K} . \tag{12}$$

In a similar fashion, the unconditional probability density function of X, Y under H_2, for hypothesized value Θ_h, is given by

$$p_2(X, Y|\Theta_h) = \left(2\pi\sigma_a\sigma_m\sigma_n\, R_3^{1/2}\right)^{-K} *$$

$$* \exp\left[-\frac{R_{an}}{2R_3\sigma_m^2}\sum_k x_k^2 - \frac{R_{am}}{2R_3\sigma_n^2}\sum_k (y_k+\Theta_h)^2 + \frac{1}{R_3\sigma_m^2\sigma_n^2}\sum_k x_k(y_k+\Theta_h)\right]. \tag{13}$$

Now if Θ_h were known, we could evaluate the likelihood ratio by taking the ratio of (10) and (13). However, we must resort instead to a generalized likelihood ratio, by computing the two values of Θ_h that maximize (10) and (13) respectively, and then taking the ratio of the two maxima [1, p. 92]. This procedure is not optimum in any sense; however, it often leads to physical processors that perform well.

The values of Θ_h that maximize (10) and (13) are given respectively by

$$\theta_1 = \frac{1}{K}\sum_k y_k + \frac{\sigma_a^2}{\sigma_a^2 + \sigma_m^2}\frac{1}{K}\sum_k x_k \quad \text{for } H_1 \ , \tag{14A}$$

$$\theta_2 = -\frac{1}{K}\sum_k y_k + \frac{\sigma_a^2}{\sigma_a^2 + \sigma_m^2}\frac{1}{K}\sum_k x_k \quad \text{for } H_2 \ , \tag{14B}$$

These results have a reasonable physical interpretation: From (5A), the sum $x(t) + y(t)$ would eliminate the random process $a(t)$, and the sample mean of the sum would give an estimate of θ under H_1. However, $m(t)$ contaminates the $a(t)$ contribution according to (3); thus the scale factor $\sigma_a^2/(\sigma_a^2 + \sigma_m^2)$ in (14A) indicates how trustworthy the sample mean of $x(t)$ is. The noise $n(t)$ in $y(t)$ is unavoidable but is partially suppressed by the inherent averaging of the sample mean of $y(t)$. A similar argument holds for $x(t) - y(t)$ under H_2.

The logarithm of the generalized likelihood ratio is (proportional to)

$$T = \frac{1}{2} R_3 \, \sigma_m^2 \, \sigma_n^2 \ln \frac{p_2\left(X, Y \middle| \theta_2\right)}{p_1\left(X, Y \middle| \theta_1\right)} =$$

$$= -\frac{1}{4} R_{am} \, \sigma_m^2 \sum_k (y_k + \theta_2)^2 + \frac{1}{2}\sum_k x_k (y_k + \theta_2) +$$

$$+ \frac{1}{4} R_{am} \, \sigma_m^2 \sum_k (y_k - \theta_1)^2 + \frac{1}{2}\sum_k x_k (y_k - \theta_1) \ , \tag{15}$$

where we used (10) and (13) with θ_1 and θ_2 substituted for θ_h. Now let

$$S_x = \sum_k x_k, \quad S_y = \sum_k y_k \ , \quad P = \sum_k x_k y_k \ . \tag{16}$$

Then (15) becomes, upon use of (14) and simplification and cancellation of various terms,

$$T = P - \frac{1}{K} S_x S_y =$$

$$= \sum_k x_k y_k - \frac{1}{K} \sum_m x_m \sum_n y_n = \tag{17A}$$

$$= \sum_k \tilde{x}_k \tilde{y}_k , \tag{17B}$$

where

$$\tilde{x}_k = x_k - \frac{1}{K} \sum_m x_m \quad \text{for } 1 \leq k \leq K ,$$

$$\tilde{y}_k = y_k - \frac{1}{K} \sum_n y_n \quad \text{for } 1 \leq k \leq K , \tag{18}$$

are defined as the sample ac components of measurements X and Y; that is, the sample means are subtracted from the measurements.

The generalized likelihood ratio test in (17) says to cross-correlate the sample ac components of both measured waveforms and to compare with zero (assuming H_1 and H_2 are equally likely apriori). That is, the test is

$$T = \sum_{k=1}^{K} \tilde{x}_k \tilde{y}_k \underset{H_1}{\overset{H_2}{\gtrless}} 0 . \tag{19}$$

Observe that this decision rule makes no use of the variances of any of the processes a(t), m(t), n(t), although this information was presumed known in the above derivation. Of course, the source angle estimates in (14) do require knowledge of the signal-to-noise ratio σ_a^2/σ_m^2 in the x(t) measurement of the antenna angle a(t). Once the decision of H_1 vs H_2 is made via (19), the corresponding estimate of the actual source angle θ is taken from (14).

On the Performance of Test (19)

From (5) and (6), we find that

$$\overline{x_k y_k} = \begin{cases} -\sigma_a^2 & \text{for } H_1 \\ \sigma_a^2 & \text{for } H_2 \end{cases} ,$$

$$\overline{x_m y_n} = 0 \quad \text{for } m \neq n \; . \tag{20}$$

Then (17A) yields the mean value of the generalized likelihood ratio test statistic as

$$T = \begin{cases} -(K-1)\,\sigma_a^2 & \text{for } H_1 \\ (K-1)\,\sigma_a^2 & \text{for } H_2 \end{cases} . \tag{21}$$

Since the statistics of T are desired different under the two hypotheses, (21) indicates that large K and σ_a^2 are desired. That is, a large observation time and a widely-moving antenna give better performance of the test; both of these conclusions are physically plausible.

Although generalized likelihood ratio test (19) does not require knowledge of any variances, the performance (in terms of the error probability) does depend on all the variances. However, the performance does not depend on the actual value θ of the source angle. To see this, we employ (5) in (18) to obtain

$$\tilde{y}_k = \begin{cases} -a_k + n_k - (-S_a + S_n)/K & \text{for } H_1 \\ a_k + n_k - (\;S_a + S_n)/K & \text{for } H_2 \end{cases} , \tag{22}$$

where

$$S_a = \sum_k a_k, \quad S_n = \sum_k n_k \ . \tag{23}$$

Thus θ is absent from (22), and since θ is not involved in $x(t)$ or x_k, test statistic T is independent of θ.

If we develop (19) in more detail and make use of (18) and (5), we find we can express

$$T_1 = -\left[\sum_k (a_k + m_k)(a_k - n_k) - \frac{1}{K}\sum_k (a_k + m_k)(a_k - n_k)\right] \text{ for } H_1 \ ,$$

$$T_2 = \left[\sum_k (a_k + m_k)(a_k + n_k) - \frac{1}{K}\sum_k (a_k + m_k)(a_k + n_k)\right] \text{ for } H_2 \ . \tag{24}$$

Thus the statistics of T_1 are identical to those of $-T_2$.

Based upon the results in [2,3], an exact analysis of the cumulative and exceedance distribution functions of test statistic (17A) is possible and will be documented in a NUSC technical report shortly; in fact, a more general processor, where the sample means term is scaled prior to subtraction, will be analyzed.

NOISELESS MEASURED ANTENNA ANGLE

In this section, the noise m(t) in the antenna angle measurement is zero; thus, from (3),

$$x(t) = a(t) \tag{25}$$

under H_1 and H_2. We also remove the Gaussian assumption on the statistics of antenna movement a(t), and allow a(t) to be completely general. Furthermore, we allow any statistical dependence among the samples A of a(t) in (7). However, we retain the Gaussian assumption on the additive noise process n(t) in (4) and (5), and keep the statistical independence of its samples $\{n(k\Delta)\}_1^K$.

Derivation of Generalized Likelihood Ratio

Based upon these premises, the conditional probability density function under H_1 of measurements X, Y, for a fixed A and hypothesized θ_h, is

$$p_1(X, Y|\theta_h, A) = \prod_{k=1}^{K} \left\{\delta(x_k - a_k) \frac{1}{\sigma_n} \phi\left(\frac{y_k - \theta_h + a_k}{\sigma_n}\right)\right\} . \tag{26}$$

(This is also the limit of (8) as $\sigma_m \to 0+$.) Then letting joint probability density function $p_a(A)$ represent the arbitrary statistical dependence of samples A, the unconditional probability density function under H_1 of X, Y is

$$p_1(X, Y|\theta_h) = \int dA \; p_a(A) \; p_1(X, Y|\theta_h, A) =$$

$$= (\sqrt{2\pi}\,\sigma_n)^{-K} \exp\left[-\frac{1}{2\sigma_n^2} \sum_k (y_k - \theta_h + x_k)^2\right] p_a(X) , \tag{27}$$

using (26), (9), and the sifting property of delta functions. (If measurement noise m(t) in (3) were non-zero, this simplification of the probability density function in (27) would not be possible.)

The value of θ_h that maximizes probability density function (27), regardless of the form of the probability density function p_a, is

$$\theta_1 = \frac{1}{K}\sum_k (x_k + y_k) \;, \tag{28}$$

which is simply the sample mean of the sum waveform x(t) + y(t). This result is consistent with the earlier one in (14A), for $\sigma_m = 0$, and the ensuing discussion.

In a similar fashion, the probability density function of X, Y under H_2 is

$$p_2(X, Y|\theta_h) = (\sqrt{2\pi}\,\sigma_n)^{-K} \exp\left[-\frac{1}{2\sigma_n^2}\sum_k (y_k + \theta_h - x_k)^2\right] p_a(X) \;, \tag{29}$$

and the maximizing choice of θ_h is

$$\theta_2 = \frac{1}{K}\sum_k (x_k - y_k) \;, \tag{30}$$

which is the sample mean of difference waveform x(t) - y(t).

There follows, from (27)-(30), the logarithm of the generalized likelihood ratio as

$$2\sigma_n^2 \,\ell n \,\frac{p_2(X, Y|\theta_2)}{p_1(X, Y|\theta_1)} = -\sum_k (y_k + \theta_2 - x_k)^2 + \sum_k (y_k - \theta_1 + x_k)^2 \;. \tag{31}$$

The generalized likelihood ratio test for the two hypotheses in (1) is therefore

$$\sum_k (x_k + y_k - \theta_1)^2 \underset{H_1}{\overset{H_2}{\gtrless}} \sum_k (x_k - y_k - \theta_2)^2 , \tag{32}$$

or upon substitution of (28) and (30), and use of (18), simply

$$\sum_k \tilde{x}_k \tilde{y}_k \underset{H_1}{\overset{H_2}{\gtrless}} 0 . \tag{33}$$

As in the previous section, the cross-correlation of the sample ac components of the measurements X, Y should be compared with zero. This decision rule holds for any statistics of antenna movement a(t).

The alternative form in (32) has an interesting interpretation: Reference to (27)-(28) reveals that θ_1 is the best constant fit to $\{x_k + y_k\}_1^K$ in a least squares sense. Thus the left side of (32) is the actual value of the least squares error of a constant fit to the sum waveform. Similarly, the right side of (32) is the least squares error of a constant fit to the difference waveform. Whichever error is smaller, that hypothesis is selected. This decision rule, (32), is consistent with the observation from (5) and (25) that

$$x(t) + y(t) = \theta + n(t) \quad \text{under } H_1 ,$$

$$x(t) - y(t) = \theta - n(t) \quad \text{under } H_2 . \tag{34}$$

That is, except for zero-mean measurement noise n(t), the sum waveform is constant under H_1, whereas the difference waveform is constant under H_2.

An alternative form for (32) and (33) is

$$T = \sum_k x_k y_k - \frac{1}{K} \sum_m x_m \sum_n y_n \mathop{\gtrless}_{H_1}^{H_2} 0 \,, \tag{35}$$

just as in (17A). All the ensuing discussion there through (24) is directly relevant for this case as well.

SUMMARY

The generalized likelihood ratio test statistic, for both the noisy as well as the noiseless antenna angle measurement, is a cross-correlator of the sample ac components of the measured antenna and source angle waveforms. Exact performance of this processor can be accomplished, since the test statistic is a quadratic form of correlated Gaussian random variables; in fact, the complete cumulative and exceedance distribution functions of the quantity

$$\sum_{k=1}^{K} x_k y_k - \frac{\gamma}{K} \sum_{m=1}^{K} x_m \sum_{n=1}^{K} y_n , \tag{36}$$

for any scaling γ, is capable of exact analysis and will be presented in a future NUSC technical report.

REFERENCES

1. H. L. Van Trees, Detection, Estimation, and Modulation Theory, Part I, J. Wiley and Sons, Inc., N.Y., 1968.

2. A. H. Nuttall, "Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly from Characteristic Functions", NUSC Technical Report 7023, 1 October 1983.

3. A. H. Nuttall, "Exact Performance of General Second-Order Processors for Gaussian Inputs", NUSC Technical Report 7035, 15 October 1983.

RESOLUTION OF AMBIGUITY FOR RANDOMLY MOVING LINE ARRAY

Albert H. Nuttall
Surface Ship Sonar Dept.
TM No. 831150
3 October 1983
UNCLASSIFIED

DISTRIBUTION LIST

EXTERNAL: CNM MAT 05B (CAPT. Z. L. NEWCOMB)
NAVSEA 63D (R. COCKERILL)
NAVSEA PMS 409-A1 (CAPT. D. BOLKA)
NAVSEA PMS 409-A11 (J. ARCHER)

P CABLE	3302	L NG	3213	
C CARTER	3314	A NUTTALL	3302	
A DUGAS	3251	N OWSLEY	3211	
R DWYER	3314	J PEARSON	3212	
E EBY	101	J PRATT	3291	
J FAY	3211	A QUAZI	3314	
J FERRIE	3212	L ROY	3521	
L FREEMAN	33	A SAENGER	3331	
B GUIMOND	3521	B SPEAR	3291	
P HERSTEIN	3331	D TAGLIAFERRI	3212	
J IANNIELLO	3212	W VON WINKLE	10	
R KNEIPFER	3212	D WILSON	3233	
R LAPLANTE	33C	J HASSAB	352 NPT	
R LATOURETTE	3213	LIBRARY GROUP, NLON	021311	(6 copies)
A LOTRING	02111	LIBRARY GROUP, NWPT	021312	(10 copies)

External: 4
Internal: 44
Total: 48

END

12-86

DTIC